



Differential Geometry II - Smooth Manifolds

Winter Term 2025/2026

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Exercise Sheet 10 – Solutions

Exercise 1:

(a) Let $F: M \rightarrow N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Show that X and Y are F -related if and only if for every smooth real-valued function f defined on an open subset of N , we have $X(f \circ F) = (Yf) \circ F$.

(b) Consider the smooth map

$$F: \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (\cos t, \sin t)$$

and the smooth vector fields

$$X = \frac{d}{dt} \in \mathfrak{X}(\mathbb{R}) \quad \text{and} \quad Y = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$

Show that X and Y are F -related.

(c) Let $F: M \rightarrow N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$. Prove that there exists a unique smooth vector field Y on N that is F -related to X . The vector field Y is denoted by F_*X and is called the *pushforward of X by F* .

(d) Consider the open submanifolds

$$M := \{(x, y) \in \mathbb{R}^2 \mid y > 0 \text{ and } x + y > 0\} \subseteq \mathbb{R}^2$$

and

$$N := \{(u, v) \in \mathbb{R}^2 \mid u > 0 \text{ and } v > 0\} \subseteq \mathbb{R}^2$$

and the map

$$F: M \rightarrow N, (x, y) \mapsto \left(x + y, \frac{x}{y} + 1\right).$$

(i) Show that F is a diffeomorphism and compute its inverse F^{-1} .

(ii) Compute the pushforward F_*X of the following smooth vector field X on M :

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x} \Big|_{(x,y)}.$$

Solution:

(a) For any point $p \in M$ and any smooth real-valued function f defined on an open neighborhood of $F(p)$ we have

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)(f)$$

and

$$((Yf) \circ F)(p) = (Yf)(F(p)) = Y_{F(p)}f.$$

Therefore, X and Y are F -related (i.e., $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$) if and only if for every smooth real-valued function f defined on an open subset of N it holds that $X(f \circ F) = (Yf) \circ F$.

(b) *1st way:* We prove the claim using the definition of F -related vector fields. To this end, recall that the differential of F at an arbitrary point $t \in \mathbb{R}$ is represented (with respect to the bases $\{d/dt|_t\}$ for $T_t\mathbb{R} \cong \mathbb{R}$ and $\{\partial/\partial x|_{F(t)}, \partial/\partial y|_{F(t)}\}$ for $T_{F(t)}\mathbb{R}^2 \cong \mathbb{R}^2$) by the Jacobian of F at t , which is the 2×1 -matrix

$$\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

Hence,

$$dF_t(X_t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \cdot (1) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = -\sin(t) \frac{\partial}{\partial x} \Big|_{F(t)} + \cos(t) \frac{\partial}{\partial y} \Big|_{F(t)} = Y_{F(t)}$$

for any $t \in \mathbb{R}$, which shows that X and Y are F -related.

2nd way: We may alternatively prove the assertion using (a) as follows: For every smooth real-valued function $f = f(x, y)$ defined on an open subset of \mathbb{R}^2 and for any $t \in \mathbb{R}$ we have

$$\begin{aligned} X(f \circ F)(t) &= X_t(f \circ F) = \frac{d}{dt} \Big|_t (f \circ F) \\ &= \left(\frac{\partial f}{\partial x}(F(t)), \frac{\partial f}{\partial y}(F(t)) \right) \cdot (F'_1(t), F'_2(t))^T \\ &= -\sin(t) \frac{\partial f}{\partial x}(F(t)) + \cos(t) \frac{\partial f}{\partial y}(F(t)) \end{aligned}$$

and

$$\begin{aligned} ((Yf) \circ F)(t) &= (Yf)(F(t)) = Y_{F(t)}f \\ &= \left(\cos(t) \frac{\partial}{\partial y} \Big|_{F(t)} - \sin(t) \frac{\partial}{\partial x} \Big|_{F(t)} \right) f \\ &= \cos(t) \frac{\partial f}{\partial y}(F(t)) - \sin(t) \frac{\partial f}{\partial x}(F(t)). \end{aligned}$$

(c) Since F is a diffeomorphism, we may define the following rough vector field on N :

$$Y: N \rightarrow TN, \quad q \mapsto dF_{F^{-1}(q)}(X_{F^{-1}(q)}).$$

It is clear that this is the unique (rough) vector field on N that is F -related to X . We now observe that Y is the composition of the following smooth maps:

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN,$$

so it is smooth by *Proposition 2.11*(d).

Remark. Given a diffeomorphism $F: M \rightarrow N$, the pushforward of any $X \in \mathfrak{X}(M)$ by F is defined explicitly by the formula

$$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}),$$

as already demonstrated in the proof of (c) above. As long as the inverse map F^{-1} of F can be computed explicitly, the pushforward of a smooth vector field can be computed directly from this formula. This observation will be applied in (d) below.

(d) It is straightforward to check that the inverse of F is given by the formula

$$F^{-1}(u, v) = \left(u - \frac{u}{v}, \frac{u}{v} \right).$$

The differential of F at an arbitrary point $(x, y) \in M$ is represented by the Jacobian of F at (x, y) , given by

$$DF(x, y) = \begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix},$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$DF \left(u - \frac{u}{v}, \frac{u}{v} \right) = \begin{pmatrix} 1 & 1 \\ \frac{v}{u} & \frac{v - v^2}{u} \end{pmatrix}.$$

For any $(u, v) \in N$ we have

$$X_{F^{-1}(u,v)} = X_{\left(u - \frac{u}{v}, \frac{u}{v}\right)} = \frac{u^2}{v^2} \frac{\partial}{\partial x} \Big|_{\left(u - \frac{u}{v}, \frac{u}{v}\right)}.$$

Therefore, we obtain

$$(F_*X)_{(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial u} \Big|_{(u,v)} + \frac{u}{v} \frac{\partial}{\partial v} \Big|_{(u,v)}.$$

Exercise 2 (to be submitted):

(a) *Coordinate formula for the Lie bracket:* Let M be a smooth n -manifold and let

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(M)$, respectively, in terms of some smooth local coordinates (x^i) for M . Show that the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^n \sum_{i=1}^n \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie brackets $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]$ of the coordinate vector fields $\partial/\partial x^i$ in any smooth chart $(U, (x^i))$ for a given smooth n -manifold M .

(c) Compute the Lie bracket of each of the following pairs of smooth vector fields on \mathbb{R}^3 :

(i) $X_1 = y \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial y}$ and $Y_1 = \frac{\partial}{\partial y}$.

(ii) $X_2 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and $Y_2 = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}$.

(iii) $X_3 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and $Y_3 = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$.

(d) Consider the open submanifold

$$M := \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \subseteq \mathbb{R}^2,$$

the map

$$F: M \rightarrow M, (x, y) \mapsto \left(xy, \frac{y}{x}\right),$$

and the smooth vector fields

$$X := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad \text{and} \quad Y := y \frac{\partial}{\partial x}$$

on M .

(i) Show that F is a diffeomorphism, compute its Jacobian matrix $DF(x, y)$ at an arbitrary point $(x, y) \in M$, and determine its inverse F^{-1} .

(ii) Compute the pushforwards F_*X and F_*Y of X and Y , respectively.

(iii) Compute the Lie brackets $[X, Y]$ and $[F_*X, F_*Y]$.

Solution:

(a) Denote by $U \subseteq M$ the coordinate domain. For any $f \in C^\infty(U)$ we have

$$\begin{aligned} [X, Y](f) &= XY(f) - YX(f) = X \left(\sum_j Y^j \frac{\partial f}{\partial x^j} \right) - Y \left(\sum_i X^i \frac{\partial f}{\partial x^i} \right) \\ &= \sum_j \left[X(Y^j) \frac{\partial f}{\partial x^j} + Y^j X \left(\frac{\partial f}{\partial x^j} \right) \right] - \sum_i \left[Y(X^i) \frac{\partial f}{\partial x^i} + X^i Y \left(\frac{\partial f}{\partial x^i} \right) \right] \\ &= \sum_{j,i} \left[X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + Y^j X^i \frac{\partial}{\partial x^i} \left(\frac{\partial f}{\partial x^j} \right) \right] - \sum_{i,j} \left[Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} + X^i Y^j \frac{\partial}{\partial x^j} \left(\frac{\partial f}{\partial x^i} \right) \right] \\ &= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^j} - \underbrace{\sum_{i,j} (X^i Y^j) \left[\frac{\partial}{\partial x^i} \left(\frac{\partial f}{\partial x^j} \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial f}{\partial x^i} \right) \right]}_{=0} \\ &= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^j}. \end{aligned}$$

(b) Recall that the component functions of each coordinate vector field $\partial/\partial x^j$ in the coordinate frame $(\partial/\partial x^i)$ associated with the smooth chart $(U, (x^i))$ are constant, so it follows immediately from (a) that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0.$$

(c) In case (i), writing

$$X_1 = 0 \frac{\partial}{\partial x} - 2xy^2 \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} \quad \text{and} \quad Y_1 = 0 \frac{\partial}{\partial x} + 1 \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial z}$$

and using part (a), we compute that

$$[X_1, Y_1] = 4xy \frac{\partial}{\partial y} - \frac{\partial}{\partial z}.$$

Similarly, in case (ii) we obtain that

$$[X_2, Y_2] = -z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z},$$

while in case (iii) we find that

$$[X_3, Y_3] = 2x \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial y}.$$

(d)(i) Define the map

$$G: M \rightarrow M, (x, y) \mapsto \left(\sqrt{x/y}, \sqrt{xy} \right).$$

Note that G is smooth, as the functions

$$(x, y) \in M \mapsto x/y \in \mathbb{R}_{>0} \quad \text{and} \quad (x, y) \in M \mapsto xy \in \mathbb{R}_{>0}$$

are smooth (they are rational polynomials with non-vanishing denominator), and the function $u \in \mathbb{R}_{>0} \mapsto \sqrt{u} \in \mathbb{R}_{>0}$ is smooth as well. Observe also that

$$(G \circ F)(x, y) = \left(\sqrt{\frac{xy}{y/x}}, \sqrt{xy \cdot \frac{y}{x}} \right) = (x, y)$$

and

$$(F \circ G)(x, y) = \left(\sqrt{\frac{x}{y}} \cdot \sqrt{xy}, \frac{\sqrt{xy}}{\sqrt{x/y}} \right) = (x, y)$$

for all $(x, y) \in M$, so F and G are mutually inverse. Hence, F is a diffeomorphism with inverse $F^{-1} = G$. Furthermore, the Jacobian of F is given by

$$DF(x, y) = \begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \quad \text{for all } (x, y) \in M.$$

(d)(ii) The push-forward F_*X of X is the unique vector field on M that is F -related to X , i.e.,

$$dF_{(x,y)}(X_{(x,y)}) = (F_*X)_{F(x,y)} \text{ for all } (x, y) \in M.$$

The matrix of $dF_{(x,y)}$ with respect to the bases provided by $\partial/\partial x$ and $\partial/\partial y$ is precisely $DF(x, y)$. Since X is given with respect to this basis by $(x, y)^T$ and since

$$\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ 0 \end{pmatrix},$$

we obtain that

$$(F_*X)_{F(x,y)} = 2xy \frac{\partial}{\partial x} \Big|_{F(x,y)}.$$

Replacing (x, y) by $G(x, y)$ yields

$$(F_*X)_{(x,y)} = 2x \frac{\partial}{\partial x} \Big|_{(x,y)}.$$

Similarly, since Y is given by $(y, 0)^T$ with respect to the standard basis and since

$$\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \cdot \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} y^2 \\ -y^2/x^2 \end{pmatrix},$$

we obtain

$$(F_*Y)_{F(x,y)} = y^2 \frac{\partial}{\partial x} \Big|_{F(x,y)} - \frac{y^2}{x^2} \frac{\partial}{\partial y} \Big|_{F(x,y)}.$$

Replacing (x, y) by $G(x, y)$ yields

$$(F_*Y)_{(x,y)} = xy \frac{\partial}{\partial x} \Big|_{(x,y)} - y^2 \frac{\partial}{\partial y} \Big|_{(x,y)}.$$

Alternatively, working as in the solution of *Exercise 1(d)* and using coordinates (u, v) in the codomain, we compute that

$$DF(G(u, v)) = \begin{pmatrix} \sqrt{uv} & \sqrt{\frac{u}{v}} \\ -\frac{\sqrt{uv}}{v} & \sqrt{\frac{v}{u}} \end{pmatrix},$$

as well as

$$X_{G(u,v)} = \sqrt{\frac{u}{v}} \frac{\partial}{\partial x} \Big|_{G(u,v)} + \sqrt{uv} \frac{\partial}{\partial y} \Big|_{G(u,v)} \quad \text{and} \quad Y_{G(u,v)} = \sqrt{uv} \frac{\partial}{\partial x} \Big|_{G(u,v)},$$

whence

$$(F_*X)_{(u,v)} = 2u \frac{\partial}{\partial u} \Big|_{(u,v)} \quad \text{and} \quad (F_*Y)_{(u,v)} = uv \frac{\partial}{\partial u} \Big|_{(u,v)} - v^2 \frac{\partial}{\partial v} \Big|_{(u,v)}.$$

(d)(iii) By part (a) we have

$$\begin{aligned}
[X, Y] &= \left(\underbrace{\left(x \frac{\partial y}{\partial x} - y \frac{\partial x}{\partial x} \right)}_{=-y} + \underbrace{\left(y \frac{\partial y}{\partial y} - 0 \cdot \frac{\partial x}{\partial y} \right)}_{=y} \right) \frac{\partial}{\partial x} + \\
&\quad + \left(\underbrace{\left(x \frac{\partial 0}{\partial x} - y \frac{\partial y}{\partial x} \right)}_{=0} + \underbrace{\left(y \frac{\partial 0}{\partial y} - 0 \cdot \frac{\partial y}{\partial y} \right)}_{=0} \right) \frac{\partial}{\partial y} \\
&= 0.
\end{aligned}$$

so by *Exercise 4(b)* we conclude that

$$[F_*X, F_*Y] = F_*[X, Y] = 0.$$

One can also check this with a direct calculation, using parts (a) and (d)(ii); namely, we have

$$\begin{aligned}
[F_*X, F_*Y] &= \left(\underbrace{\left(2x \frac{\partial(xy)}{\partial x} - xy \frac{\partial(2x)}{\partial x} \right)}_{=0} + \underbrace{\left(0 \cdot \frac{\partial(xy)}{\partial y} + y^2 \frac{\partial(2x)}{\partial y} \right)}_{=0} \right) \frac{\partial}{\partial x} \\
&\quad + \left(\underbrace{\left(2x \frac{\partial(-y^2)}{\partial x} - xy \frac{\partial 0}{\partial x} \right)}_{=0} + \underbrace{\left(0 \cdot \frac{\partial(-y^2)}{\partial y} + y^2 \cdot \frac{\partial 0}{\partial y} \right)}_{=0} \right) \frac{\partial}{\partial y} \\
&= 0.
\end{aligned}$$

Exercise 3 (*Properties of the Lie bracket*): Let M be a smooth manifold. Show that the Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$\begin{aligned}
[aX + bY, Z] &= a[X, Z] + b[Y, Z], \\
[Z, aX + bY] &= a[Z, X] + b[Z, Y].
\end{aligned}$$

(b) *Antisymmetry*:

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(d) For all $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

Solution:

(a) We first make the following observation: given $\lambda, \mu \in \mathbb{R}$ and $U, V, W \in \mathfrak{X}(M)$, for any $f \in C^\infty(M)$ it holds that

$$(\lambda V + \mu W)Uf = \lambda VUf + \mu WUf \quad \text{and} \quad U(\lambda V + \mu W)f = \lambda UVf + \mu UWf.$$

Indeed, for any $p \in M$ we have

$$\begin{aligned} ((\lambda V + \mu W)Uf)(p) &= (\lambda V + \mu W)_p(Uf) = (\lambda V_p + \mu W_p)(Uf) \\ &= \lambda V_p(Uf) + \mu W_p(Uf) = \lambda V(Uf)(p) + \mu W(Uf)(p) \\ &= (\lambda VUf + \mu WUf)(p), \end{aligned}$$

which yields the first equality above, while the second one is obtained analogously.

Now, given $a, b \in \mathbb{R}$, using the previous observation, for any $f \in C^\infty(M)$ we have

$$\begin{aligned} [aX + bY, Z](f) &= (aX + bY)Zf - Z(aX + bY)f \\ &= aXZf + bYZf - aZXf - bZYf \\ &= a(XZf - ZXf) + b(YZf - ZYf) \\ &= a[X, Z](f) + b[Y, Z](f) \\ &= (a[X, Z] + b[Y, Z])(f), \end{aligned}$$

which yields the first part of the statement, while the second one is obtained similarly.

(b) For any $f \in C^\infty(M)$ we have

$$[X, Y](f) = XYf - YXf = -(YXf - XYf) = -[Y, X](f),$$

which yields the statement.

(c) By expanding all the brackets and using linearity we obtain

$$\begin{aligned} [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] &= \\ &= X[Y, Z] - [Y, Z]X + Y[Z, X] - [Z, X]Y + Z[X, Y] - [X, Y]Z \\ &= XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + \\ &\quad + ZXY - ZYX - XYZ + YXZ \\ &= 0. \end{aligned}$$

(d) We first make the following observation: if $V \in \mathfrak{X}(M)$ and $s, t \in C^\infty(M)$, then

$$(sV)h = s(Vh) \quad (\text{as smooth functions on } M),$$

since for any $p \in M$ we have

$$((sV)h)(p) = (sV)_p h = (s(p)V_p)h = s(p)V_p h = s(p)(Vh)(p) = (s(Vh))(p).$$

Now, fix $f, g \in C^\infty(M)$. Using the previous observation and the fact that smooth vector fields are derivations of $C^\infty(M)$ by *Proposition 7.6*, for any $h \in C^\infty(M)$ we have

$$\begin{aligned}
[fX, gY](h) &= (fX)(gY)(h) - (gY)(fX)(h) \\
&= (fX)(g(Yh)) - (gY)(f(Xh)) \\
&= g(fX)(Yh) + (Yh)(fX)(g) - f(gY)(Xh) - (Xh)(gY)(f) \\
&= gf(X(Yh)) + f(Xg)(Yh) - fg(Y(Xh)) - g(Yf)(Xh) \\
&= fg((XY - YX)(h)) + (fXg)Y(h) - (gYf)X(h) \\
&= (fg[X, Y] + (fXg)Y - (gYf)X)(h),
\end{aligned}$$

whence the desired relation.

Remark. A *Lie algebra* (over \mathbb{R}) is an \mathbb{R} -vector space \mathfrak{g} endowed with a map $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called the *Lie bracket* and usually denoted by $(X, Y) \mapsto [X, Y]$, which satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

(a) *Bilinearity:* For all $a, b \in \mathbb{R}$ we have

$$\begin{aligned}
[aX + bY, Z] &= a[X, Z] + b[Y, Z], \\
[Z, aX + bY] &= a[Z, X] + b[Z, Y].
\end{aligned}$$

(b) *Antisymmetry:*

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

According to *Exercise 3*, the infinite-dimensional \mathbb{R} -vector space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket. Here are two more examples of Lie algebras:

- (1) The \mathbb{R} -vector space $M_n(\mathbb{R})$ of real $n \times n$ matrices equipped with the *commutator bracket* $[A, B] := AB - BA$ becomes an n^2 -dimensional Lie algebra, which is denoted by $\mathfrak{gl}(n, \mathbb{R})$.
- (2) If V is an \mathbb{R} -vector space, then the \mathbb{R} -vector space of all linear maps from V to itself equipped with the *commutator bracket* $[L_1, L_2] := L_1 \circ L_2 - L_2 \circ L_1$ becomes a Lie algebra, which is denoted by $\mathfrak{gl}(V)$.

Under our usual identification of $n \times n$ matrices with linear maps from \mathbb{R}^n to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

Exercise 4: Let $F: M \rightarrow N$ be a smooth map.

- (a) *Naturality of the Lie bracket:* Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be smooth vector fields such that X_i is F -related to Y_i for $i \in \{1, 2\}$. Show that $[X_1, X_2]$ is F -related to $[Y_1, Y_2]$.
- (b) *Pushforwards of Lie brackets:* Assume that F is a diffeomorphism and consider $X_1, X_2 \in \mathfrak{X}(M)$. Show that $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.

Solution:

(a) Since X_i is F -related to Y_i for $i \in \{1, 2\}$, by *Exercise 1(a)* we infer that for every smooth real-valued function f defined on an open subset of N we have

$$X_1(f \circ F) = (Y_1 f) \circ F \quad \text{and} \quad X_2(f \circ F) = (Y_2 f) \circ F.$$

Therefore,

$$\begin{aligned} [X_1, X_2](f \circ F) &= X_1 X_2(f \circ F) - X_2 X_1(f \circ F) \\ &= X_1((Y_2 f) \circ F) - X_2((Y_1 f) \circ F) \\ &= (Y_1(Y_2 f)) \circ F - (Y_2(Y_1 f)) \circ F \\ &= ([Y_1, Y_2](f)) \circ F, \end{aligned}$$

and *Exercise 1(a)* implies now that $[X_1, X_2]$ is F -related to $[Y_1, Y_2]$.

(b) Follows immediately from part (a) and from *Exercise 1(c)*.

Exercise 5:

(a) *Restricting smooth vector fields to submanifolds:* Let M be a smooth manifold, let S be an immersed submanifold of M , and let $\iota: S \hookrightarrow M$ be the inclusion map. Prove the following assertions:

- (i) If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is ι -related to Y , then $Y \in \mathfrak{X}(M)$ is tangent to S .
- (ii) If $Y \in \mathfrak{X}(M)$ is tangent to S , then there is a unique smooth vector field on S , denoted by $Y|_S$, which is ι -related to Y .

[Hint: Determine first the candidate vector field on S and then use *Theorem 5.9* and *Proposition 5.20* to show that it is smooth.]

(b) *Lie brackets of smooth vector fields tangent to submanifolds:* Let M be a smooth manifold and let S be an immersed submanifold of M . If Y_1 and Y_2 are smooth vector fields on M that are tangent to S , then show that their Lie bracket $[Y_1, Y_2]$ is also tangent to S .

Solution:

(a)(i) Since X is ι -related to Y , it holds that $Y_p = d\iota_p(X_p)$ for all $p \in S$, which means that $Y_p \in T_p S$ for all $p \in S$, i.e., Y is tangent to S .

(a)(ii) Since by hypothesis we have $Y_p \in T_p S \cong d\iota_p(T_p S)$ for all $p \in S$, we may define a rough vector field $X: S \rightarrow TS$ by requiring that, for any $p \in S$, its value $X_p \in T_p S$ is the unique vector such that $d\iota_p(X_p) = Y_p$. By the injectivity of $d\iota_p$, it is clear that X is unique, and that it is ι -related to Y , so it remains to show that X is smooth. To this end, let $p \in S$ be arbitrary. By *Proposition 5.20* there is an open neighborhood V of p in S such that V is embedded in M . By *Theorem 5.9* there exists a smooth chart $(U, (x^i))$ for M such that $V \cap U$ is a k -slice in U – we may assume that $V \cap U$ is the k -slice given by $x^{k+1} = \dots = x^n = 0$ – and (x^1, \dots, x^k) are local coordinates for S in $V \cap U$. Consider the coordinate representation

$$Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$$

of Y on U . Since Y is tangent to S , by *Proposition 7.9* (evaluating the above expression at the coordinate function x^i with $i > k$) we infer that $Y^{k+1} = \dots = Y^n = 0$ on $V \cap U$. Therefore,

$$X = \sum_{1 \leq i \leq k} Y^i \Big|_{U \cap V} \frac{\partial}{\partial x^i} \Big|_{U \cap V}$$

is the coordinate representation of X on $V \cap U$, and each component $Y^i|_{U \cap V}$ is smooth by [*Exercise Sheet 7, Exercise 5(a)*], so X is smooth on $U \cap V$ by *Proposition 7.2*, and we are done.

(b) Since $Y_1, Y_2 \in \mathfrak{X}(M)$ are tangent to S , by part (a)(ii) there exist (unique) smooth vector fields W_1, W_2 on S such that W_j is ι -related to Y_j for each $j \in \{1, 2\}$, where $\iota: S \hookrightarrow M$ is the inclusion map. But now *Exercise 4(a)* implies that $[W_1, W_2] \in \mathfrak{X}(S)$ is ι -related to $[Y_1, Y_2] \in \mathfrak{X}(M)$, and thus $[Y_1, Y_2]$ is tangent to S by part (a)(i).